

THE KAKEYA MAXIMAL OPERATOR ON THE VARIABLE LEBESGUE SPACES

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ABSTRACT. We shall verify the Kakeya (Nikodym) maximal operator K_N , $N \gg 1$, is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^2)$ when the exponent function $p(\cdot)$ is N -modified locally log-Hölder continuous and log-Hölder continuous at infinity.

1. INTRODUCTION

The purpose of this paper is to investigate the boundedness of the Kakeya (Nikodym) maximal operator on the variable Lebesgue spaces. Given a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, we define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of measurable functions such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} = \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ generalizes the classical Lebesgue space $L^p(\mathbb{R}^n)$: if $p(\cdot) \equiv p_0$, then $L^{p(\cdot)}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n)$. Variable Lebesgue spaces have been studied in the past twenty years (see [1, 3, 4, 6, 7, 8, 9, 13, 14, 15]). For a locally integrable function f on \mathbb{R}^n the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in Q \in \mathcal{Q}} \int_Q |f(y)| dy,$$

where we have used \mathcal{Q} to denote the family of all cubes in \mathbb{R}^n with sides parallel to the coordinate axes and $\int_Q f(x) dx$ to denote the usual integral average of f over Q . Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions $p(\cdot)$ for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. By the classical Hardy-Littlewood maximal theorem, any constant function $p(\cdot) \equiv p_0$ with $1 < p_0 < \infty$ belongs to $\mathcal{P}(\mathbb{R}^n)$. In [7], L. Diening showed that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists a positive constant c such that for any family of pairwise disjoint cubes π and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\left\| \sum_{Q \in \pi} \int_Q |f(y)| dy \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where χ_E stands for the characteristic function of a measurable set $E \subset \mathbb{R}^n$. This result implies, for example, that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p'(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, where $p'(x) = \frac{p(x)}{p(x)-1}$. However, since this result is very general, some simple sufficient conditions for which $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ has

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been studied by many authors (see [6, 5, 14, 15]). In [5], D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer give a new and simpler proof of the boundedness of the Hardy-Littlewood maximal operator M on variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 1.1. (a): We say that $p(\cdot)$ is locally log-Hölder continuous if there exists a positive constant c_0 such that

$$|p(x) - p(y)| \log \left(\frac{1}{|x - y|} \right) \leq c_0, \quad x, y \in \mathbb{R}^n, |x - y| < 1.$$

(b): We say that $p(\cdot)$ is log-Hölder continuous at infinity if there exist constants c_∞ and $p(\infty)$ such that

$$|p(x) - p(\infty)| \log(e + |x|) \leq c_\infty, \quad x \in \mathbb{R}^n.$$

(c): Given a measurable set $E \subset \mathbb{R}^n$, let

$$p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x) \text{ and } p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x).$$

If $E = \mathbb{R}^n$, then we simply write p_- and p_+ .

Proposition 1.2 ([5, Theorem 1.2]). *Let $1 < p_- \leq p_+ < \infty$. Suppose that $p(\cdot)$ is locally log-Hölder continuous and log-Hölder continuous at infinity. Then there exists a positive constant C independent of f such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

For a locally integrable function f on \mathbb{R}^2 the Kakeya (Nikodym) maximal operator K_N , $N \gg 1$, is defined by

$$K_N f(x) = \sup_{x \in R \in \mathcal{B}_N} \int_R |f(y)| dy,$$

where \mathcal{B}_N denotes the set of all rectangles in \mathbb{R}^2 with eccentricity N (the ratio of the length of long-sides and short-sides is equal to N). In this paper, we investigate the boundedness property of the Kakeya maximal operator K_N on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^2)$. It is well known that (see [2, 10, 16])

$$(1.1) \quad \|K_N f\|_{L^p(\mathbb{R}^2)} \leq C_p (\log N)^{2/p} \|f\|_{L^p(\mathbb{R}^2)} \text{ for } 2 \leq p \leq \infty.$$

One might naturally expect that

$$\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)} \leq C (\log N)^{2/p_-} \text{ when } 2 \leq p_- \leq p_+ < \infty,$$

where $\|T\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)}$ denotes the operator norm $T : L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)$. However, we have the following theorem.

Theorem 1.3. *Let $N \gg 1$ and $1 < p_- < p_+ < \infty$. Suppose that K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$ and that $p(\cdot)$ is continuous. Then there exist a positive constant C , independent of N , and a small constant $\varepsilon > 0$ such that*

$$\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)} \geq CN^\varepsilon.$$

Thus, in the framework of the variable Lebesgue spaces, we are interested in a small positive constant c such that N^c bounds from above $\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)}$.

The main result of this paper is the following (Theorem 1.5). The technique of the proof of our theorem is due to [3], which is used the machinery of Calderón-Zygmund cubes. We apply this technique to the rectangles in \mathcal{B}_N . For the precise estimate we need the following notion.

Definition 1.4. Let $N \gg 1$. We say that $p(\cdot)$ is N -modified locally log-Hölder continuous if there exists a positive constant c_N such that

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \log \left(\frac{N}{|x-y|^2} \right) \leq c_N \log N, \quad x, y \in \mathbb{R}^2, |x-y| < \sqrt{N}.$$

Theorem 1.5. Let $N \gg 1$ and $2 \leq p_- \leq p_+ < \infty$. Suppose that $p(\cdot)$ is N -modified locally log-Hölder continuous and log-Hölder continuous at infinity. Then K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$ and

$$\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)} \leq CN^{p_- - c_N} (\log N)^{2/p_-},$$

where the constant C is independent of N .

Corollary 1.6. Let $N \gg 1$ and $2 \leq p_- \leq p_+ < \infty$. Suppose that $p(\cdot)$ is locally log-Hölder continuous and log-Hölder continuous at infinity. Then K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$ and

$$\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \rightarrow L^{p(\cdot)}(\mathbb{R}^2)} \leq CN^{p_- - C_N} (\log N)^{2/p_-},$$

where the constant C is independent of N and

$$C_N = \sup_{\substack{x, y \in \mathbb{R}^2 \\ |x-y| < \sqrt{N}}} \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right|.$$

Remark 1.7. We remark that

$$p_- C_N \leq p_- \left(\frac{1}{p_-} - \frac{1}{p_+} \right) = 1 - \frac{p_-}{p_+}.$$

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2 , do not change in different occurrences.

2. PROOF OF THEOREM 1.3

The following argument is due to T. Kopaliani [12] (see also [11]). Recall that the conjugate function $p'(x)$ is defined by $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. The following generalized Hölder inequality and a duality relation can be found in [13]:

$$\int_{\mathbb{R}^2} |f(x)g(x)| dx \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

$$\|f\|_{p(\cdot)} \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_{\mathbb{R}^2} |f(x)g(x)| dx.$$

Suppose that K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$. Then for every rectangle $R \in \mathcal{B}_N$ we have

$$\|K_N\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \geq \|K_N f\|_{p(\cdot)} \geq \left\| \int_R f(y) dy \chi_R \right\|_{p(\cdot)} = \int_R f(y) dy \|\chi_R\|_{p(\cdot)}$$

for all nonnegative f with $\|f\|_{p(\cdot)} \leq 1$. Taking supremum all such f , we have

$$(2.1) \quad \|K_N\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \geq \frac{1}{|R|} \|\chi_R\|_{p'(\cdot)} \|\chi_R\|_{p(\cdot)}$$

for all $R \in \mathcal{B}_N$, where $|R|$ denotes the area of the rectangle R .

Suppose that $p(\cdot)$ is continuous and is not constant. Then we can find two closed squares B_1 and B_2 in \mathbb{R}^2 with $|B_1|, |B_2| < 1$, such that

$$(2.2) \quad p_+(B_1) < p_-(B_2).$$

Without loss of generality we may assume that

$$B_1 = [0, s] \times [0, s] \text{ and } B_2 = [0, s] \times [t - s, t] \text{ for some } t > s > 0.$$

We take N with $t/N < s$ and let $R = [0, t/N] \times [0, t]$. Then we have $R \in \mathcal{B}_N$ and

$$|R \cap B_1| = |R \cap B_2| = \frac{st}{N}.$$

Observe now that the following embeddings hold:

$$\begin{aligned} L^{p(\cdot)}(B_2) &\hookrightarrow L^{p_-(B_2)}(B_2), \\ L^{p'(\cdot)}(B_1) &\hookrightarrow L^{(p_+(B_1))'}(B_1), \end{aligned}$$

where $\frac{1}{(p_+(B_1))'} + \frac{1}{p_+(B_1)} = 1$. It follows that

$$\begin{aligned} \frac{1}{|R|} \|\chi_R\|_{p(\cdot)} \|\chi_R\|_{p'(\cdot)} &\geq \frac{1}{|R|} \|\chi_{R \cap B_2}\|_{L^{p(\cdot)}(B_2)} \|\chi_{R \cap B_1}\|_{L^{p'(\cdot)}(B_1)} \\ &\geq \frac{1}{|R|} \|\chi_{R \cap B_2}\|_{L^{p_-(B_2)}(B_2)} \|\chi_{R \cap B_1}\|_{L^{(p_+(B_1))'}(B_1)} \\ &= |R|^{-1} \cdot |R \cap B_2|^{\frac{1}{p_-(B_2)}} \cdot |R \cap B_1|^{1 - \frac{1}{p_+(B_1)}} \\ &= t^{-2} \cdot (st)^{1 + \frac{1}{p_-(B_2)} - \frac{1}{p_+(B_1)}} \cdot N^{\frac{1}{p_+(B_1)} - \frac{1}{p_-(B_2)}}, \end{aligned}$$

where we have used $|B_1|, |B_2| < 1$. Since by (2.2) $\frac{1}{p_+(B_1)} - \frac{1}{p_-(B_2)} > 0$, we conclude by (2.1) that $\|K_N\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}}$ has a lower bound N^ε with $\varepsilon > 0$.

3. PROOF OF THEOREM 1.5

In what follows we shall prove Theorem 1.5. We need two lemmas.

Lemma 3.1. *Let $N \gg 1$. Suppose that $p(\cdot)$ is N -modified locally log-Hölder continuous. Then, for any rectangle $R \in \mathcal{B}_N$,*

$$|R|^{\frac{1}{p_+(R)} - \frac{1}{p_-(R)}} \leq N^{c_N}.$$

Proof. When $|R| \geq 1$, there is nothing to prove. Suppose that $|R| < 1$. Since $p(\cdot)$ is continuous, there exist $x, y \in R$ such that $p(x) = p_-(R)$ and $p(y) = p_+(R)$. It follows that

$$\begin{aligned} |R|^{\frac{1}{p_+(R)} - \frac{1}{p_-(R)}} &= |R|^{\frac{1}{p(y)} - \frac{1}{p(x)}} \leq \left(\frac{|x - y|^2}{N} \right)^{\frac{1}{p(y)} - \frac{1}{p(x)}} \\ &= \exp \left\{ \left(\frac{1}{p(y)} - \frac{1}{p(x)} \right) \log \left(\frac{|x - y|^2}{N} \right) \right\} = \exp \left\{ \left(\frac{1}{p(x)} - \frac{1}{p(y)} \right) \log \left(\frac{N}{|x - y|^2} \right) \right\} \\ &\leq \exp \{ \log(N^{c_N}) \} = N^{c_N}, \end{aligned}$$

where we have used $|x - y| < \sqrt{N}$ and the N -modified local log-Hölder continuity of $p(\cdot)$. \square

Lemma 3.2 ([3, Lemma 2.4]). *Suppose that $p(\cdot)$ is log-Hölder continuous at infinity. Let $P(x) = (e + |x|)^{-M}$, $M \geq 2$. Then there exists a constant c depending on M , $p(\infty)$ and c_∞ such that given any set E and any function F such that $0 \leq F(y) \leq 1$, $y \in E$,*

$$\begin{aligned} \int_E F(y)^{p(y)} dy &\leq c \int_E F(y)^{p(\infty)} dy + c \int_E P(y)^{p(\infty)} dy, \\ \int_E F(y)^{p(\infty)} dy &\leq c \int_E F(y)^{p(y)} dy + c \int_E P(y)^{p(\infty)} dy. \end{aligned}$$

Proof of Theorem 1.5. We may assume that f is nonnegative. We first linearize the operator K_N . For $k \in \mathbb{N}$, we denote by \mathcal{D}_k the family of all dyadic cubes $Q = 2^{-k}(m + [0, 1]^2)$, $m \in \mathbb{Z}^2$. For each $Q \in \mathcal{D}_k$ we choose a rectangle $R(Q) \in \mathcal{B}_N$, such that $R(Q) \supset Q$. We denote the operator T_k as

$$T_k f(x) = \sum_{Q \in \mathcal{D}_k} \int_{R(Q)} f(y) dy \chi_Q(x).$$

By definition it is easy to see that

$$T_k f(x) \leq K_N f(x)$$

for any choice of rectangles $\{R(Q)\}$. On the other hand, there is a sequence of linearized operators $\{T_k f\}$ which converge pointwise to $K_N f$ as k tends to infinity. Thus, by the Fatou theorem we need only prove Theorem 1.5 with K_N replaced by T_k with a constant C not depending on k .

By homogeneity we may assume that $\|f\|_{p(\cdot)} = 1$. Then

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^2} f(x)^{p(x)} dx \leq 1.$$

Decompose f as $f_1 + f_2$, where $f_1 = f \chi_{\{x: f(x) > 1\}}$ and $f_2 = f \chi_{\{x: f(x) \leq 1\}}$. Then

$$\rho_{p(\cdot)}(f_i) \leq 1 \text{ for } i = 1, 2.$$

Let

$$C_1 = \sup_{R \in \mathcal{B}_N} |R|^{\frac{1}{p_+(R)} - \frac{1}{p_-(R)}},$$

$$C_2 = (\log N)^{2/p_-}.$$

The estimate for f_1 . We shall verify that, if $\lambda_1 = C_1^{p_-} C_2$, then

$$(3.1) \quad \rho_{p(\cdot)}\left(\frac{T_k f_1}{\lambda_1}\right) = \int_{\mathbb{R}^2} \left(\frac{T_k f_1(x)}{\lambda_1}\right)^{p(x)} dx \leq C.$$

It follows from Hölder's inequality that

$$\begin{aligned} & \rho_{p(\cdot)}\left(\frac{T_k f_1}{\lambda_1}\right) \\ &= \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_1}\right)^{p(x)} \left(\int_{R(Q)} f_1(y) dy\right)^{p(x)} dx \\ &\leq \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_1}\right)^{p(x)} \left(\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}} dy\right)^{\frac{p_-(x)}{p_-(R(Q))}} dx \\ &= \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_1}\right)^{p(x)} \left(\frac{1}{|R(Q)|}\right)^{\frac{p_-(x)}{p_-(R(Q))}} \left(\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}} dy\right)^{\frac{p_-(x)}{p_-(R(Q))}} dx. \end{aligned}$$

There holds, for $|R(Q)| \geq 1$,

$$\left(\frac{1}{C_1^{p_-}}\right)^{p(x)} \left(\frac{1}{|R(Q)|}\right)^{\frac{p_-(x)}{p_-(R(Q))}} \leq \left(\frac{1}{|R(Q)|}\right)^{p_-},$$

where we have used $C_1 \geq 1$ and $\frac{p(x)}{p_-(R(Q))} \geq 1$. Also, there holds, for $|R(Q)| < 1$,

$$\begin{aligned}
& \left(\frac{1}{C_1^{p_-}} \right)^{p(x)} \left(\frac{1}{|R(Q)|} \right)^{\frac{p-p(x)}{p_-(R(Q))}} \\
& \leq \left(\frac{1}{|R(Q)|} \right)^{\frac{p-p(x)}{p_+(R(Q))} - \frac{p-p(x)}{p_-(R(Q))}} \left(\frac{1}{|R(Q)|} \right)^{\frac{p-p(x)}{p_-(R(Q))}} \\
& = \left(\frac{1}{|R(Q)|} \right)^{\frac{p-p(x)}{p_+(R(Q))}} = \left(\frac{1}{|R(Q)|} \right)^{\frac{p-p(x)}{p_+(R(Q))} - p_-} \left(\frac{1}{|R(Q)|} \right)^{p_-} \\
& \leq \left(\frac{1}{|R(Q)|} \right)^{p_-},
\end{aligned}$$

where we have used

$$C_1 \geq |R(Q)|^{\frac{1}{p_+(R(Q))} - \frac{1}{p_-(R(Q))}} \text{ and } \frac{p(x)}{p_+(R(Q))} \leq 1.$$

We see that by the definition of f_1

$$\begin{aligned}
& \left(\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}} dy \right)^{\frac{p-p(x)}{p_-(R(Q))}} \\
& \leq \left(\int_{R(Q)} f_1(y)^{p(y)} dy \right)^{\frac{p-p(x)}{p_-(R(Q))} - p_-} \left(\int_{R(Q)} f_1(y)^{\frac{p(y)}{p_-}} dy \right)^{p_-} \\
& \leq \left(\int_{R(Q)} f_1(y)^{\frac{p(y)}{p_-}} dy \right)^{p_-},
\end{aligned}$$

where we have used

$$\left(\int_{R(Q)} f_1(y)^{p(y)} dy \right)^{\frac{p-p(x)}{p_-(R(Q))} - p_-} \leq \left(\int_{\mathbb{R}^2} f(y)^{p(y)} dy \right)^{\frac{p-p(x)}{p_-(R(Q))} - p_-} \leq 1.$$

These yield

$$\rho_{p(\cdot)} \left(\frac{T_k f_1}{\lambda_1} \right) \leq \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{C_2} \right)^{p(x)} \left(\int_{R(Q)} f_1(y)^{\frac{p(y)}{p_-}} dy \right)^{p_-} dx.$$

Therefore, since $R(Q) \supset Q$ and $\frac{p(x)}{p_-} \geq 1$,

$$\begin{aligned}
\rho_{p(\cdot)} \left(\frac{T_k f_1}{\lambda_1} \right) & \leq \frac{1}{(\log N)^2} \int_{\mathbb{R}^2} K_N[f_1^{p(\cdot)/p_-}](x)^{p_-} dx \\
& \leq C \int_{\mathbb{R}^2} f_1(x)^{p(x)} dx \leq C,
\end{aligned}$$

where we have used (1.1).

The estimate for f_2 . We shall verify that, if $\lambda_2 = C_2$, then

$$(3.2) \quad \rho_{p(\cdot)} \left(\frac{T_k f_2}{\lambda_2} \right) = \int_{\mathbb{R}^2} \left(\frac{T_k f_2(x)}{\lambda_2} \right)^{p(x)} dx \leq C.$$

Since $f_2 \leq 1$, we immediately see that

$$F = \frac{1}{\lambda_2} \int_{R(Q)} f_2(y) dy \leq 1.$$

Therefore, by Lemma 3.2, with $P(x) = (e + |x|)^{-2}$,

$$\begin{aligned} \rho_{p(\cdot)}\left(\frac{T_k f_2}{\lambda_2}\right) &= \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_2} \int_{R(Q)} f_2(y) dy \right)^{p(x)} dx \\ &\leq C \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_2} \int_{R(Q)} f_2(y) dy \right)^{p(\infty)} dx + C \sum_{Q \in \mathcal{D}_k} \int_Q P(x)^{p(\infty)} dx. \end{aligned}$$

Since $p(\infty) \geq 2$ and the cubes $Q \in \mathcal{D}_k$ are disjoint, we can immediately estimate the second term:

$$\sum_{Q \in \mathcal{D}_k} \int_Q P(x)^{p(\infty)} dx = \int_{\mathbb{R}^2} P(x)^{p(\infty)} dx \leq C.$$

We shall estimate the first term. It follows that

$$\begin{aligned} &\sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda_2} \int_{R(Q)} f_2(y) dy \right)^{p(\infty)} dx \\ &\leq \frac{1}{(\log N)^2} \sum_{Q \in \mathcal{D}_k} \int_Q K_N f_2(x)^{p(\infty)} dx \\ &\leq C \int_{\mathbb{R}^2} f_2(x)^{p(\infty)} dx, \end{aligned}$$

where we have used (1.1). Since $f_2 \leq 1$ we can apply Lemma 3.2 again,

$$\int_{\mathbb{R}^2} f_2(x)^{p(\infty)} dx \leq C \int_{\mathbb{R}^2} f_2(x)^{p(x)} dx + C \int_{\mathbb{R}^2} P(x)^{p(\infty)} dx \leq C.$$

Altogether, we obtain (3.2).

Conclusion. The estimates (3.1), (3.2) and Lemma 3.1 yield the theorem.

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